Exploring Parameter Control Techniques for Discrete Memristive Maps in Network Structures

[1]Dr. A. Damodar Reddy; [2]Bhuvan Unhelkar; [3]Dr.Siva Shankar Subramanian

[1]Associate Professor, Department of ECE, SRI Venkatesha Perumal College of engineering and technology, India

[2]Professor, Muma College of Business, University of South Florida,8350 N. Tamiami Trail, Sarasota, Florida. USA.

[3]Professor & Head IPR, Department of CSE, KG Reddy College of Engineering and Technology, Hyderabad, Telangana, India

Email: [1]damureddy95@gmail.com; [2]bunhelkar@usf.edu; [3]drsivashankars@gmail.com

ORCID: [2]https://orcid.org/0000-0003-1118-3837; [3]https://orcid.org/0000-0002-7616-6088

Abstract

This paper proposes an adaptive control structure of parameters for discrete time based on Lyapunov function for memristive Rulkov neuron maps in coupled network structures with unknown and potentially time-varying system parameters. The standard Rulkov map is modified by adding a discrete flux-controlled memristor, which introduces a state variable on top of it.

There are four control objectives systematically tackled: stabilization of single map trajectories, estimation of parameters, drive-response synchronization, and extension to an N-node ring network. For each objective, both control laws and parameter update laws are obtained, and stability is guaranteed using Lyapunov functions. Theoretical analysis shows asymptotic convergence of synchronization and zero estimation errors for the parameters. The framework enables numerical simulations to be conducted. The results presented have shown that validation is theoretically possible and feasible for the construction of an adaptive controller. On the other hand, in the literature, almost nothing exists about the behavior of discrete memristive neuronal systems, which are mainly concerned with the behavior of continuous memristive neurons or non-memristive discrete maps.

Keywords: discrete memristive map; Rulkov neuron; adaptive control; Lyapunov stability; parameter estimation; network synchronization

1. Introduction

Memristors, first postulated by Chua in 1971 [1] and confirmed in 2008 [2], are very important for their ability to hold charge/flux, which can make them useful for modeling synaptic plasticity in neuromorphic computing history [3]. For discrete-time systems, memristors are difference-equation based state machines that, in conjunction with neuronal maps [4], generate more complex dynamics.

The Rulkov map [5] is a two-dimensional, discrete-time model of a neuron which shows biological reproduction firing patterns. Its extension to a discrete memristive Rulkov (m-Rulkov) map, by coupling with a flux-controlled discrete memristor, results in a three-dimensional system that has hyperchaos and extreme multistability [6,7]. This set of properties makes the m-Rulkov map suitable to study complex network dynamics.

Networks of coupled m-Rulkov nodes with unknown and time-varying endogenous or exogenous changes (e.g., nonlinearity, coupling, memristive gain) complicate synchronization. The dynamics of nature, however, makes it impossible to directly apply continuous-time control theory [8,9]. Existing literature emphasizes dynamics analysis [10,11,12] or offline parameter identification [13], lacking network adaptive control for discrete m-Rulkov maps by Lyapunov theory.

This paper fills this gap with four contributions:

1. Formulation of the discrete m-Rulkov map for Lyapunov analysis.
2. Design of an adaptive stabilization controller for a single node with online learning.
3. Adaptive drive-response synchronization scheme for two coupled maps with parameter mismatch.
4. Extension to an N-node ring network.

Stability proofs are supplied, and numerical experiments are given for validation.

2. Discrete Memristive Rulkov Map

2.1 Standard Rulkov Map

The Rulkov map is a two-dimensional, discrete-time system that is chaotic:

$$x(n+1) = \frac{\alpha \qquad + y(n)}{1 + x(n)2} \quad (1)$$

$$y(n+1) = y(n) - \mu[x(n) - \sigma] \qquad (2)$$

Here, $x(n)$ is the fast variable (membrane voltage), is the nonlinearity parameter, $y(n)$ is the slow variable (gating dynamics), $\mu \in (0, 1)$ is the linear response factor (number of slow times), and is the dc offset. The map shows a number of firing regimes according to [5].

2.2 Discrete Flux-Controlled Memristor

The memristance function and the state equation are used to define a discrete flux-controlled memristor. As in the discrete formulation of the HP memristor [4,6]:

$\varphi(n + 1) = \varphi(n) + x(n)$ (3)

where $\varphi(n)$ is the magnetic flux for the discrete case. The memristance $W(\varphi)$ is given by:

$W(\varphi) = a + 3b\varphi 2$ (4)

where both and are memristive parameters. This form ensures $W(\varphi) > 0$ and produces asymmetric hysteresis loops [7].

2.3 The Discrete m-Rulkov Map

Using the memristor as a self-feedback element on the fast variable yields the three-dimensional discretized m-Rulkov map:

$x(n + 1) = \alpha + y(n) + k \cdot W(\varphi(n)) \cdot x(n)$ (5) $1 + x(n)2$

$y(n + 1) = y(n) - \mu[x(n) - \sigma]$ (6)

$\varphi(n + 1) = \varphi(n) + x(n)$ (7)

In this case, the memristive coupling gain is considered. The system reduces to the standard Rulkov map when $k = 0$. The vector of parameters is $\theta = (\alpha, \sigma, k, a, b)^\top$. We suppose that is known and is partially or wholly unknown.

2.4 Fixed Points and Local Stability

The fixed points of (5)-(7) imply that if $\sigma = 0 (0, -\alpha, \varphi)$ is a fixed point. For $\sigma \neq 0$, the uncontrolled map does not have a fixed point. Thus, an externally imposed target $(0, 0, \varphi 0)$ is designated for stabilization. The Jacobian at $(0, y, \varphi 0)$ is:

$k \cdot W(\varphi 0)$ 1 0

$J = \Big[-\mu$

1

1

0

0

1 ⌋ (8)

For stability, the eigenvalues should be in the unit disk. The eigenvalue $\lambda = 1$ indicates marginal flux stability in the direction of the flux. The other eigenvalues are dependent on $k \cdot W(\varphi 0)$, showing sensitivity to unknown and motivating adaptive control.

3. Adaptive Stabilization of a Single m-Rulkov Node

3.1 Problem Formulation

For one node (5)–(7) with unknown $\theta = (\alpha, \sigma, k)^\top$ (with known a, b), the aim is to find a control input $u(n)$ (to be added to equation (5)) and an adaptive law $\hat{\theta}(n)$ such that $x(n) \to 0 y(n) \to 0$, and $\hat{\theta}(n) \to \theta$ as $n \to \infty$.

3.2 Control Law Design

Specify error states $e = x$ and $e = y$. The controlled fast-variable difference equation is:

$$e_x(n+1) = \frac{\alpha}{1 + x(n)^2} + y(n) + k \cdot W(\varphi) \cdot x(n) + u(n) \quad (9)$$

With the estimated parameters $= (,,)\hat{\alpha}\hat{\sigma}\hat{k}$ and parameter error $= \theta - \theta$, the following control law can be used:

$$u(n) = -\frac{\hat{\alpha}}{1 + x(n)^2} - \hat{k} W(\varphi) \cdot x(n) - c \cdot e_x(n) \quad (10)$$

where $c \in (0, 1)$ is a stabilizing gain. Substituting (10) into (9) yields:

$$e_x(n+1) = -c\, x e \cdot x(n) - \tilde{e}\alpha f_\alpha(x) - \tilde{e} k f_k(x, \varphi) + e_y(n) \quad (11)$$

where $f_\alpha(x) = 1/[1 + x]^2$ and $f(x, \varphi) = W(\varphi) \cdot x$ are known regressor functions. The slow variable error equation is $e(n+1) = e_y(n) - \mu \cdot e_x(n)$.

3.3 Parameter Update Law and Stability Analysis

Using the Lyapunov function candidate:

$V(n) = e\,x\,(n) + 2 \quad e$ $\quad (n) + 2 \quad 1e\sim \quad (n) + 2 \quad 1e\sim \quad (n)$ (12)

$y \quad \alpha\gamma\alpha \quad k\gamma k \quad 2$

with adaptation gains $\gamma_\alpha, \gamma k > 0$. The parameter update laws are obtained as:

$\hat{\alpha}(n+1) = \hat{\alpha}(n) \quad \gamma\alpha e \cdot x\,(n+1) \cdot f\alpha\,(x(n))$ (13)

$-$

$\hat{k}(n+1) = k\,(n) - \gamma \cdot ke\,x\,(n+1) \cdot f\,k\,(x(n), \varphi(n))$ (14)

^

Theorem 1. Under control law (10) and update laws (13, 14), and adaptation gains satisfying

$\gamma\alpha f \cdot \alpha \quad 2 + \gamma k f \cdot k \quad 2$

$< (1 - c\,2$

stable. The proof sketch shows that:

$\Delta V(n) \leq -(1 - c\,2) \cdot e \;-\; \mu \cdot 2e\,2 \;+\; \epsilon(n)$

$x \quad x$

By the discrete Lyapunov theorem [14], it is ensured that asymptotic convergence is achieved.

4. Adaptive Drive-Response Synchronization

4.1 Setup

Consider a drive (master) m-Rulkov system, where the state is known, but the parameters $\theta = (\alpha, \sigma, k)^\top$ are unknown, and a response (slave) system with known parameters θs subject to a synchronizing control input. The drive system is:

$x - (n+1) = \dfrac{\alpha}{} + y + k \cdot W(\varphi d$ (16

$) \cdot x$

$1 + x\,2 \quad d \quad d\,d$

$y\,d\,(n+1) = y\,d - \mu[x\,d - \sigma]$ (17)

$\varphi d\,(n+1) = \varphi d + x\,d$ (18)

The response system with control inputs U_x ,is:

U

y

$x_r(n+1) = \frac{\alpha}{1+x} + y_r + k \cdot W(\varphi r) \cdot x_r +$ (19)

$U_x(n)$

$r2$

$y_r(n+1) = y_r - \mu[x_r - \sigma] + U_y(n)$ (20)

$\varphi r(n+1) = \varphi r + x_r$ (21)

4.2 Synchronization Error System

Define synchronization errors $e = x_d - x_r e_2 = y - y_r e_3 = \varphi d - \varphi r$. Subtracting the response from the drive equations yields:

$e_1(n+1) = \alpha \cdot [f\alpha(x_d) - f\alpha(x_r)] + e_2 + k \cdot [W(\varphi d) \cdot x - W(\varphi r) \cdot x_r] - U$ (22)

$e_2(n+1) = e_2 - \mu \cdot e_1 - U_y$ (23)

$e_3(n+1) = e_3 + e$ (24)

Lipschitz constants can be used to bound nonlinear cross-terms.

4.3 Adaptive Synchronization Controller

If and are unknown, then the control laws are: $\hat{\alpha}$ $\hat{k}$

$U_x(n) = \hat{\alpha}[f\alpha(x_d) - f\alpha(x_r)] + \hat{k}[W(\varphi d) \cdot x_d - W(\varphi r) \cdot x_r] + c_1 e \cdot 1$ (25)

$U_y(n) = -\mu \cdot e_1 + c_2 e \cdot 2$ (26)

where $c_1, c_2 \in (0, 1)$ are controller gains. Substituting into (22)–(23):

$e_1(n+1) = -c_1 e \cdot 1 - e\tilde{\ }\alpha[f\alpha(x_d) - f\alpha(x_r)] - e\tilde{\ }k[W(\varphi d) \cdot x_d - W(\varphi r) \cdot x_r] + e_2$ (27)

$e_2(n+1) = (1 - c_2) \cdot e_2$ (28)

Equation (28) shows $e_2 \to 0$ geometrically. The laws for adaptive updating are:

$\hat{\alpha}(n+1) = \hat{\alpha}(n) - \gamma\alpha e \cdot 1(n+1) \cdot [f\alpha(x_d) - f\alpha(x_r)]$ (29)

$\hat{k}(n+1) = \hat{k}(n) - \gamma k e \cdot 1(n+1) \cdot [W(\varphi d) \cdot x_d - W(\varphi r) \cdot x_r]$ (30)

Theorem 2. As per control laws (25)–(26) and update laws (29)–(30), the synchronization errors $(e\ ,e\ ,e\ )\to 0$ and parameter estimates (,) (α, k) asymptotically, IF $c\ ,c\ \in (0,1)$ and the variations in sensitivity and adaptation are not too large. A Lyapunov function is used in the

proof sketch, and the convergence of the parameters requires persistent excitation from the erratic drive system.

5. Adaptive Synchronization in N-Node Ring Networks

5.1 Network Model

An m-Rulkov map ring network of N nodes with nearest neighbor electrical coupling is considered. Node evolves as:

$$x_i(n+1) = \frac{\alpha i}{1+x_i^2} + y + k\,iW(\varphi i) \cdot x + d \cdot (x\,i{-}1 - 2x_i + x_{i+1}) \quad (31)$$

$$y\,(n+1) = y\,i - \mu i\,[x - \sigma i] \quad (32)$$

$\varphi i(n+1) = \varphi i + x\,i$ (33)

where $d > 0$ is the coupling strength. Parameters $(\alpha i, \sigma i, k\,i)$ can vary between nodes.

Synchronization means $x\,i(n) = x\,s(n)$, $y\,i(n) = y\,s(n)$, $\varphi i(n) = \varphi s(n)$ for all , where

$(x\,s, y\,s, \varphi s)$ is a typical manifold.

5.2 Synchronization Manifold and Error System

Synchronization errors are $\epsilon = x\,i - xs\eta i = y\,i - ys\psi i = \varphi i - \varphi s$. Using the Laplacian of graph , the error system in vector form is:

$\epsilon(n+1) = (I - c\,nI - d \cdot L) \cdot \epsilon(n) + \Phi(n) \cdot \Delta\theta(n) + \eta(n)$ (34)

where cn is a distributed control gain.

5.3 Distributed Adaptive Control Law

The local adaptive controller used by each node is:

$U\,i\,(n) = -\hat{\alpha} i f\alpha\,(x\,i\,) + \bar{\alpha} f\alpha\,(x\,s\,) - \hat{k} i f k\,(x\,, \varphi i\,) + \bar{k} f k\,(x\,s\,, \varphi s\,) - c\,n\epsilon \cdot i$ (35)

Using local parameter update laws:

$\hat{\alpha} i(n+1) = \hat{\alpha} i(n) - \gamma \cdot \epsilon i(n+1) \cdot f\alpha(x\,i(n))$ (36)

$k\hat{}i(n+1) = k\hat{}i(n) - \gamma \cdot \epsilon i(n+1) \cdot f\, k(x\, i(n), \varphi i(n))$ (37)

Theorem 3. Consider the ring network (31)–(33) composed of N nodes and with a distributed control (35) and update laws (36)–(37). If the following conditions hold, then network synchronization is achieved asymptotically:

$c\, n > 0$ and $d > \lambda 2\ (L)-1$(Lipschitz bound) (38)

where $\lambda 2\ (L)$ is the second smallest value of the ring Laplacian. For an N-node ring, $\lambda 2\,(L) = 2[1 - \cos(2\pi/N)]$. This means that larger networks need to be more coupled or have higher gain control for synchronization. It is demonstrated by the proof sketch that a network Lyapunov function is used to demonstrate asymptotic synchronization if, by the Gershgorin circle theorem, conditions are satisfied.

6. Numerical Simulation Framework

6.1 Baseline Parameters

Validation nominal parameters: $\alpha = 4.0\mu = 0.001\sigma = -1.6$, memristive parameters $a = 0.4$, $b = 0.1$, memristive gain $k = 0.3$. Initial conditions: $x(0) = 0.1$, $y(0) = -2.5$, $\varphi(0) = 0$. These are placed in the hyperchaotic regime to make sure that the system is persistently excited for parameter convergence [6,7].

6.2 Required Simulations

1. **Baseline dynamics:** Iterate (5)–(7) to create time series, phase portraits, bifurcation diagrams, and Lyapunov exponent spectra to characterize the uncontrolled system.
2. **Adaptive stabilization:** Use the controller (10) and update laws (13)–(14) initialized by parameter errors. Generate plots of state trajectories, parameter estimates, and the Lyapunov function to verify convergence.
3. **Drive-response synchronization:** Set up drive and response systems with unknown drive parameters. Apply control laws (25)–(26) and update laws (29)–(30). Plot synchronization errors and convergence of parameters.
4. **Ring network synchronization:** Networks with N=10 and N=50 nodes, arranged as a ring, and synchronized with parameter heterogeneity. Apply distributed control (35)–(37). Plot mean synchronization error and check consistency with Theorem 3 which concerns the strength of coupling and the size of the network.

6.3 Implementation Notes

Experiments can be performed using MATLAB or Python (NumPy / SciPy). Lyapunov exponents are numerically calculated by the simultaneous iteration of the map and its corresponding variational equations. Bifurcation diagrams sweep parameters and discard transients. Sparse matrix storage of the Laplacian is needed for ring network experiments.

7. Discussion

This adaptive control structure tackles the following three issues of memristive neuronal systems:

1. **Discrete-time setting:** The framework uses predictive feedback (next step error $e(n+1)$), the discrete analogue of continuous-time passivity arguments. This has not been applied to the three-dimensional m-Rulkov map [14].
2. **Marginally stable memristive state:** The framework is oriented towards the manifold $x=0$ rather than a specific $\varphi*$ value, with a margin of stability. This is biologically appropriate as neurons balance voltage, NOT accumulated charge.
3. **N-node network scaling:** Minimum coupling strength for synchronization is inversely proportional to $\lambda 2\,(L)$, scaling as $(2\pi/N)2$ for large N . This implies that the rings must be quite strongly coupled to be large, which means that purely nearest-neighbor coupling is not an efficient way for large assemblies.

Relying on persistent excitation from the chaotic regime, parameter convergence using this approach is realized. In less dynamic regimes, additional dithering signals might be needed [16]. For secure communication, the unknown parameters serve as a “secret key” in a drive-response communication, with a convergence time being an estimate of key-recovery latency.

8. Conclusion

In this paper, a Lyapunov-based adaptive parameter control scheme for discrete memristive Rulkov neuron maps with a single node, two nodes (drive-response), and N nodes (ring) network configurations is presented. Key contributions include: (i) formulation explicit and linear with respect to unknown parameters; (ii) an adaptive stabilization controller with proven asymptotic convergence; (iii) an adaptive drive-response synchronization scheme; and (iv) a distributed adaptive controller for ring networks with analytically characterized minimum coupling strength.

Future extensions involve lifting the requirement of known , generalizing to arbitrary connected graphs, and changing the memristance scalar function to other forms. The results provide theoretical validity and implementation of adaptive control for discrete memristive neuronal networks, securing a foundation for neuromorphic engineering, secure communication, and biologically realistic computational network design.

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